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LETTER TO THE EDITOR

A non-Gaussian single-mode squeezed state of the simple harmonic oscillator

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Abstract. A unitary operator U_R, which generates a single-mode squeezed state of the simple harmonic oscillator from the vacuum, is discussed. The states generated by U_R are of the form $\psi_0(\alpha) \sim \Sigma_n \alpha^n |n\rangle$. $\psi_0(\alpha)$ are not minimum-uncertainty states but have a small value for the uncertainty product even for large values of the mean number of quanta $\bar{n}$; for example, $\mathrm{var}(q)\,\mathrm{var}(p) \sim 0.74$ for $\bar{n} = 10^6$.

Squeezed coherent states of the electromagnetic field have been discussed extensively in the literature in quantum optics [1-4]. Several reports on the successful experimental generation and detection of squeezed light have appeared [5-8]. The possibility of detecting squeezed states of other bosonic systems has also been raised recently [9]. The quantum states of the simple harmonic oscillator (SHO) are relevant not only in the analysis of the electromagnetic field but also in the study of molecular, nuclear and solid state systems. The discussion of production and detection of squeezed states of the oscillator has relevance in many areas of physics.

Theoretical attention on single-mode squeezed states of light has until now centred on Gaussian wavepacket solutions of the Schrödinger equation for the SHO. It is well known that the coherent states and the squeezed coherent states are Gaussian and are the only possible minimum-uncertainty states of the SHO [4]. The mechanism for the detection of squeezed states in the laboratory, however, does not depend upon whether the state under observation is a minimum-uncertainty state or not. It is therefore possible to extend the discussion of squeezed states beyond Gaussian wavepackets. In this letter, states of the SHO which are not minimum-uncertainty states but have, nevertheless, squeezed variances for certain operators and are generated from the vacuum by the application of a unitary operator, are discussed. Planck's constant $\hbar$ and the frequency ω of the SHO are set equal to 1 for the sake of simplicity of notation.

A single mode of a free bosonic system may be described by the creation and annihilation operators a^+ and a of the SHO which satisfy the commutation relation $[a, a^+] = 1$. The Hamiltonian for the SHO is $H = N + \frac{1}{2}$ in which $N = a^+a$ is the number operator. The oscillator energy eigenstates are denoted by $|n\rangle$, $n = 0, 1, 2, \ldots$, where $N|n\rangle = n|n\rangle$. The Hermitian linear combinations

$$q = \frac{1}{\sqrt{2}}(a + a^+) \qquad p = -\frac{\mathrm{i}}{\sqrt{2}}(a - a^+) \tag{1}$$

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satisfy the commutation relation $[q, p] = i$. In the absence of interactions, a pure state of the form

$$\psi = \sum_n b_n |n\rangle \left(\sum_n |b_n|^2 \right)^{-1/2} \tag{2}$$

will evolve in time as

$$\psi(t) = \sum_n b_n \exp[-i(n+\tfrac{1}{2})t] |n\rangle \left(\sum_n |b_n|^2 \right)^{-1/2}. \tag{3}$$

If the coefficients b_n are of the form

$$b_n = C_n \exp(in\varphi) \qquad C_n = |b_n| \tag{4}$$

then the definition of the variance of an operator X in the state ψ given by

$$\mathrm{var}(X) = \langle\psi|X^2|\psi\rangle - \langle\psi|X|\psi\rangle^2 \tag{5}$$

and the relations

$$a|n\rangle = \sqrt{n}|n-1\rangle \qquad a^+|n\rangle = \sqrt{n+1}|n+1\rangle \tag{6}$$

may be used to give

$$\mathrm{var}(q) = \tfrac{1}{2} - F + G\cos^2(t-\varphi) \tag{7}$$

and

$$\mathrm{var}(p) = \tfrac{1}{2} - F + G\sin^2(t-\varphi) \tag{8}$$

where

$$F = \left(\sum_{n=0}^{\infty} \{C_n C_{n+2}[(n+1)(n+2)]^{1/2} + C_{n+1}^2(n+1)\} \right) \left(\sum_n C_n^2 \right)^{-1} \tag{9}$$

and

$$G = \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_n C_m \{C_{n+2} C_m [(n+1)(n+2)]^{1/2} - C_{m+2} C_n [(m+1)(m+2)]^{1/2} - 2C_{n+1} C_{m+1} [(n+1)(m+1)]^{1/2}\} \right) \left(\sum_n C_n^2 \right)^{-2}. \tag{10}$$

The coherent states ψ_c correspond to the choice $C_n = \alpha^n/\sqrt{n!}$, for which F and G vanish and the variances of q and p are time independent and have the value $\frac{1}{2}$. ψ_c is a minimum-uncertainty Gaussian wavepacket. The squeezed coherent states ψ_s are also Gaussian wavepackets. The Gaussian squeezed vacuum state ψ_{S0} corresponds to the choice of coefficients

$$C_{2m} = (-\nu/\mu)^m (2m-1)!!/\sqrt{(2m)!} \qquad C_{2m+1} = 0 \qquad m = 0, 1, 2, \ldots$$

with $(\mu^2 - \nu^2) = 1$. This choice of coefficients gives $F = -\nu(\nu+\mu)$ and $G = -2\nu\mu$. Hence

$$\mathrm{var}(q)_{S0} = \tfrac{1}{2}(\mu^2 + \nu^2 - 2\mu\nu\cos 2t) \tag{11}$$

and

$$\mathrm{var}(p)_{S0} = \tfrac{1}{2}(\mu^2 + \nu^2 + 2\mu\nu\cos 2t). \tag{12}$$

It is evident that for $t = n\pi/2$, $n = 0, 1, 2, \ldots$, the product of the variances of q and p equals $\frac{1}{4}$, which is the value of the variance product for a minimum-uncertainty wavepacket. The time evolution of the variances of q and p shows that at any instant of time the variance of one of the coordinates (q, p) can dip below $\frac{1}{2}$, which is the value of the variance for a minimum-uncertainty state, and will periodically rise above $\frac{1}{2}$ when the variance of the conjugate coordinate begins to fall below $\frac{1}{2}$ so that the uncertainty relation $\mathrm{var}(q)\,\mathrm{var}(p) \geq \frac{1}{4}$ is not violated. It is the possibility for the variance to dip below the value for the ground state of the SHO, which is a minimum-uncertainty state, that gives the squeezed coherent states their importance. The experimental method for the detection of squeezed light is the homodyne detection scheme [3, 10]. In this mechanism for observing squeezing the magnitude of the variance of the conjugate partner to the coordinate under observation seems to be irrelevant and it seems immaterial whether the product of the variance equals $\frac{1}{4}$ or not.

It is possible to extend the discussion of squeezing beyond squeezed coherent states and consider non-Gaussian states of the SHO that might also exhibit squeezing of the variance of an operator. In the infinite parameter space spanned by the set of coefficients b_n there may be many choices of values of b_n that could lead to the squeezing of the variance of an operator. However, among all such possible squeezed states those that can be shown to be generated by a unitary operator deserve special consideration because the existence of a unitary operator can often lead to a possible mechanism for generating that state from the vacuum. For example, it is the unitary time evolution of the vacuum under the influence of appropriate interaction Hamiltonians which makes it possible to produce coherent and squeezed coherent states of light in the laboratory. Motivated by such reasoning, a unitary operator, which generates a squeezed state from the vacuum, will now be considered.

For the SHO it is possible to define operators [11, 12] R and R^+ such that

$$R|n\rangle = (n)|n-1\rangle \qquad R^+|n\rangle = (n+1)|n+1\rangle. \tag{13}$$

R and R^+ may be related to Hermitian operators S and C [13–15], whose expectation values for coherent states correspond in the large $\langle N\rangle$ limit to $\sin\theta$ and $\cos\theta$, respectively, θ being the phase of the classical field. The operators S and C may be written in the form

$$S = (P - P^+)/2\mathrm{i} \qquad C = (P + P^+)/2 \tag{14}$$

where

$$P|n\rangle = |n-1\rangle \qquad n \neq 0$$

$$P|0\rangle = 0 \qquad P^+|n\rangle = |n+1\rangle. \tag{15}$$

R and R^+ are related to S, C, P and P^+ by

$$R = PN = (N+1)P = a\sqrt{N} = \sqrt{N+1}\,a \tag{16}$$

and

$$R^+ = NP^+ = P^+(N+1) = \sqrt{N}\,a^+ = a^+\sqrt{N+1}. \tag{17}$$

Operation on the number states enables the establishment of the equivalence of the different ways of representing R and R^+.

It is easy to show that R, R^+ and $(N+\frac{1}{2})$ satisfy the commutation relations of the SU(1, 1) algebra

$$[R, N+\tfrac{1}{2}]=R \qquad [R^+, N+\tfrac{1}{2}]=-R^+ \qquad [R, R^+]=2(N+\tfrac{1}{2}). \tag{18}$$

From R and R^+ the unitary operator

$$U_R=\exp(\gamma\, e^{i\delta}R^+-\gamma\, e^{-i\delta}R) \tag{19}$$

may be constructed. By using standard methods [16, 17], U_R may be simplified to

$$U_R=[\exp(R^+\, e^{i\delta}\tanh\gamma)](\operatorname{sech}\gamma)^{2N+1}[\exp(-R\, e^{-i\delta}\tanh\gamma)]. \tag{20}$$

The state generated by the action of U_R upon the vacuum is given by

$$\psi_0=U_R|0\rangle=(\operatorname{sech}\gamma)\sum_{n=0}^{\infty}(e^{i\delta}\tanh\gamma)^n|n\rangle. \tag{21}$$

The pure state ψ_0 of the SHO is a linear superposition of number states with coefficients arranged in a geometric progression. ψ_0 is an eigenstate of a generalised annihilation operator. Generalised annihilation and creation operators B and B^+ analogous to those defined for the squeezed coherent states may be identified through a Bogoliubov transform and calculated using standard techniques [18] to give

$$B=U_RRU_R^+=R\cosh^2\gamma+R^+\sinh^2\gamma\, e^{2i\delta}-(2N+1)\, e^{i\delta}\sinh\gamma\cosh\gamma \tag{22}$$

and

$$B^+=U_RR^+U_R^+=R^+\cosh^2\gamma+R\sinh^2\gamma\, e^{-2i\delta}-(2N+1)\, e^{-i\delta}\sinh\gamma\cosh\gamma. \tag{23}$$

It is clear that

$$BU_R|n\rangle=(n)U_R|n-1\rangle \tag{24}$$

and

$$B^+U_R|n\rangle=(n+1)U_R|n+1\rangle. \tag{25}$$

The transformed operators B and B^+ satisfy the commutation relation

$$[B, B^+]=(2N+1)\cosh 2\gamma-(e^{-i\delta}R+e^{i\delta}R^+)\sinh 2\gamma. \tag{26}$$

Using (20) and (25), B^+ may be successively applied to generate the state

$$|\psi_m\rangle=U_R|m\rangle=\frac{1}{m!}(B^+)^m|\psi_0\rangle. \tag{27}$$

ψ_m is a state with m generalised quanta and is explicitly given by

$$\psi_m=\operatorname{sech}\gamma\sum_{n=0}^{\infty}(e^{i\delta}\tanh\gamma)^{n-m}\left(\sum_{j=0}^{k}(\operatorname{sech}^2\gamma)^j(-\tanh^2\gamma)^{m-j}\frac{n!}{(n-j)!j!}\frac{m!}{(m-j)!j!}\right)|n\rangle \tag{28}$$

where k is the smaller of (m, n).

It is easy to establish that ψ_0 is a squeezed state. Equations (3), (4), (9) and (21) show that

$$F=\operatorname{sech}^2\gamma\sum_{n=0}^{\infty}(\tanh\gamma)^{2n+2}\{[(n+1)(n+2)]^{1/2}-(n+1)\}. \tag{29}$$

F is positive definite since the terms in curly brackets always give positive contributions. Examination of (7) and (8) shows that the positivity of F guarantees that as a function of time either $\mathrm{var}(q)$ or $\mathrm{var}(p)$ can dip below $\frac{1}{2}$. Such a squeezing occurs for all values of γ. Hence ψ_0 is a squeezed vacuum state.

The expectation value of an operator σ, which is diagonal in the number space, is given by

$$\langle\psi_0|\sigma|\psi_0\rangle = \mathrm{sech}^2\,\gamma \sum_{n=0}^{\infty} (\tanh\gamma)^{2n}\langle n|\sigma|n\rangle. \tag{30}$$

This expectation value equals the thermal average calculated with a density matrix of the form

$$\rho = (1-\mathrm{e}^{-\beta}) \sum_{n=0}^{\infty} \mathrm{e}^{-n\beta}|n\rangle\langle n| \qquad \beta = 1/KT \tag{31}$$

if

$$\mathrm{e}^{-\beta} = \tanh^2\gamma. \tag{32}$$

This equivalence holds for any diagonal operator σ. In particular, for the number operator N the variance in the state ψ_0 is given by

$$\mathrm{var}(N) = \bar{n}(\bar{n}+1) \tag{33}$$

where $\bar{n}$, the mean number of quanta, is related to γ by

$$\bar{n} = \langle\psi_0|N|\psi_0\rangle = \sinh^2\gamma. \tag{34}$$

Because of the equivalence of this variance to a thermal average, the number statistics in the pure state ψ_0 is indistinguishable from that for a thermal density matrix. Similar behaviour has been noted in the discussion of parametric interactions that give rise to two-mode squeezing [19] and also in a discussion of thermofield analysis of squeezing [18]. However, for operators which are not purely diagonal in the number space, the expectation values in the pure state ψ_0 are different from those for a thermal density matrix. For example, for the non-diagonal phase-related operators S and C the variances in the state ψ_0 can be shown to have the time-independent value given by

$$\mathrm{var}(S) = \mathrm{var}(C) = \tfrac{1}{4}(\bar{n}+1)^{-1} \tag{35}$$

while for a thermal density matrix

$$\mathrm{var}(S)_{\mathrm{T}} = \mathrm{var}(C)_{\mathrm{T}} = \tfrac{1}{4}(2\bar{n}+1)(\bar{n}+1)^{-1}. \tag{36}$$

It is possible to estimate the extent of squeezing of the state ψ_0 as a function of γ. Equation (8) shows that the minimum value of the variance of p is given by

$$\mathrm{var}(p)_{\min} = \tfrac{1}{2} - F. \tag{37}$$

Using (29) and (34), it is possible to show that

$$\mathrm{var}(p)_{\min} = \tfrac{1}{2}\frac{1}{(\bar{n}+1)}\bigg(1+\tfrac{1}{4}\ln(\bar{n}+1) - 2\sum_{m=1}^{\infty}[\bar{n}/(\bar{n}+1)]^m\{[m(m+1)]^{1/2}-(m+\tfrac{1}{2})+(1/8m)\}\bigg). \tag{38}$$

For $\bar{n} > 1$ accurate estimates of $\mathrm{var}(p)_{\min}$ can be made using the first few terms of the series. The asymptotic behaviour for large $\bar{n}$ is given by

$$\lim_{\bar{n}\to\infty} \mathrm{var}(p)_{\min} = (1/8\bar{n}) \ln \bar{n} \tag{39}$$

showing that $\mathrm{var}(p)_{\min} \to 0$ for $\bar{n} \to \infty$. For a squeezed coherent state the corresponding limiting value is given by

$$\lim_{\bar{n}\to\infty} (\mathrm{var}(p)_{\min})_{\mathrm{S}} = 1/8\bar{n}. \tag{40}$$

For $\bar{n} \gg 1$ it is also possible to develop an asymptotic formula for the sum of the variances in the form

$$\mathrm{var}(p) + \mathrm{var}(q) = (2\bar{n}+1)(1-\tfrac{1}{4}\pi) + \mathrm{O}(\bar{n}^{-2}). \tag{41}$$

The proofs of (38) and (41) will be given in a future publication. For squeezed coherent states the variance sum equals $(2\bar{n}+1)$ for all values of $\bar{n}$. It can be shown that for all values of $\bar{n}$ the sum of the variances of q and p in the state ψ_0 is less than that for squeezed coherent states with the same $\bar{n}$. From (38) and (41) it is easily shown that

$$\lim_{\bar{n}\to\infty} \mathrm{var}(p)\,\mathrm{var}(q) = \tfrac{1}{4}(1-\tfrac{1}{4}\pi) \ln \bar{n} \tag{42}$$

which is a slowly varying function of $\bar{n}$. The variances of p and q may be numerically evaluated for all values of γ. The variance product equals $\frac{1}{4}$ for $\gamma = 0$ and begins to rise slowly as γ increases. Equation (42) shows that even for $\bar{n} = 10^6$ the variance product is only 0.74, which is approximately three times the value for the minimum-uncertainty wavepackets. Such a low value of the variance product for such a high value of $\bar{n}$ together with the squeezing of variances for all values of $\bar{n}$ clearly shows that the states $\psi_0(\gamma)$ belong to a special category. Among all possible linear combinations of number states, the Gaussian wavepackets have a special status because of their well known properties. The states $\psi_0(\gamma)$ generated by U_R also belong to a special class because of their properties reported in this letter.

To summarise, it has been shown that a simple wavepacket made up of a superposition of number states with coefficients in a geometric progression can be generated by the application of a unitary operator on the vacuum state. The state ψ_0 so produced exhibits squeezing of (q, p) for all values of the mean number of quanta. The number statistics for the state ψ_0 is indistinguishable from that for a suitably chosen thermal density matrix. However, for non-diagonal operators such as q and p and the phase-related operators S and C, the variances in the pure state ψ_0 are different from those for a thermal density matrix. Even though $\psi_0(\gamma)$ are not minimum-uncertainty states, the uncertainty product is quite small even for very large values of the mean boson number. A full analysis of the results reported here will be presented in a future publication.

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